The Stone Theorem on Pairwise Paracompactness of Quasi-metrizable Spaces

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***Abstract*-The semantic values of programs or the data stored in a computer are usually rather discrete. Metrics can be used to study how these values relate to each other. In addition, we sometimes use a metric or generalizations of it to help us understand how discrete data stored inside a computer relate to the continuous real word. Paracompactness is an important topological property of metric spaces as it extends the benefits of finiteness[1] to infinite sets. In this paper, A. H. Stone's Theorem, that every metric space is paracompact, is extended to quasi-metric spaces. Generally, inter alia, paracompactness in asymmetric topological spaces is related to locally finite topological spaces, which play an important role in digital topology and network science.**

Keywords- Quasi-pseudometrizable Space; Locally Finite (Spaces); Pairwise Paracompact

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I. INTRODUCTION

Metric spaces and domain theory constitute fundamental mathematical tools in computer science. The metric space approach to semantics is at odds with the approach via Scott domains, simply because metric spaces are necessarily Hausdorff, whereas the natural topology of a domain, in which there are elements representing partial information, is non-Hausdorff (see [1]). If we seek a common framework for the two approaches, an obvious suggestion is to try quasi-metrics. The concept of a quasi-metric space has some interesting applications in computer science. For example, several interesting hyper and function spaces that are used in computable analysis are not symmetric and hence not metrizable (see [2, 3]). On the other hand, these spaces are typically T_0-spaces with countable bases and thus at least quasi-metrizable. In fact, there are a lot of applications of quasi-metric spaces to computer science, abstract languages and the analysis of the complexity of programs; see, for instance, [4-8].

Paracompactness is an important topological property, which plays a key role in the Nagata-Smirnov-Bing solution of the general metrization problem as well as in the theory of manifolds because: (i) it is a sufficient condition on a space to construct a partition of unity; (ii) a manifold is metrizable if and only if it is paracompact. Dieudonne in [9] proved that every separable metrizable topological space is paracompact, and conjectured that this remains true without separability. Stone [10] proved that this conjecture is true.

The fact that a quasi-metric gives rise to a conjugate quasi-metric was noticed by Kelly [11], thus leading to the study of bitopological spaces. Since Kelly [11] initiated the study of bitopological spaces, several authors have considered the problem of defining pairwise paracompactness for such spaces. A lot of definitions for pairwise paracompactness in bitopological spaces have been given, which in the case of topological spaces coincide with the usual notion of paracompactness. The problem with all these definitions is the absence of a natural extension of Stone's Theorem, that every metrizable topological space is paracompact in bitopological spaces. In the present paper, we provide a non-symmetric version of paracompactness of metric spaces.

The non-symmetrical version of paracompactness, inter alia, plays an important role in digital topology and network science. More precisely, T_0 Alexandroff spaces[2] have been studied as topological models of the supports of digital images. By the bitopological view of [12, Theorem 2.8(4)] we conclude that paracompact Alexandroff spaces are locally finite spaces.[3] The study of locally finite spaces has been extensively used in the digital topology and network science. This happens, since, a locally finite space can be locally represented in a computer, which enables one to perform topological investigations, experiments and calculations by means of computers.

II. RESULTS

A quasi-pseudometric space is a pair (X, d) such that X is a non-empty set and d is a non-negative real-valued function on $X \times X$ such that for all x, y, z $\in$ X: (i) d(x, x) =0, and (ii) d(x, y) $\leq$ d(x, z) + d(z, y).

[1]Any topology on a finite set is paracompact, but a finite set is usually considered to be a discrete topological space. Here is why: (i) if a set D has a discrete topology, then D is paracompact if and only if D is finite; (ii) a topology on a finite set D is Hausdorff if and only if it is the discrete topology; (iii) any function from a space with the discrete topology is continuous.

[2]An Alexandroff space is a topological space such that every point has a minimal neighborhood (see [13]).

[3]A topological space (X, τ) is called locally finite if each element of X has a finite neighborhood.

The function d is called a quasi-pseudometric. If d satisfies the additional condition, (iii) $d(x, y) = 0$ implies that $x = y$, then d is a quasi-metric. A metric space is a quasi-metric space (X, d) in such a way that d satisfies the next additional condition (iv) for all x, y ∈ X, d(x, y) = d(y, x). In that case, the function d is called a metric. The conjugate of a quasi-metric d on X is the quasi-metric d^{-1} given by d^{-1}(x, y) = d(y, x). Each quasi-metric d on X generates a topology τ_d on X which has the family of open d-balls $\{B_d(x, \varepsilon)|x \in X, \varepsilon > 0\}$ as base, where $B_d(x, \varepsilon)=\{y \in X|d(x, y)<\varepsilon\}$ for all x ∈ X and ε > 0. A bitopological space (X,τ_1,τ_2) is a set X together with two topologies τ_1 and τ_2 of X. The space (X,τ_1,τ_2) is quasi-metrizable if there is a quasi-metric d on X compatible with (X,τ_1,τ_2), where d is compatible with (X,τ_1,τ_2) provided that $\tau_d=\tau_1$ and $\tau_{d^{-1}}=\tau_2$.

Definition 2.1. (See [14]). A Cover C of a bitopological space (X,τ_1,τ_2) is defined to be *pairwise open* if C is a subset of $\tau_1 \cup \tau_2$ and contains nonempty sets from both τ_1 and τ_2.

Definition 2.2. (See [15, Definition 2.2]). A pairwise open cover C of a bitopological space (X,τ_1,τ_2) is said to be a *parallel refinement* of a pairwise open cover $\mathcal{U}$ of X, if for each i ∈ {1, 2}, every τ_i-open set of C is contained in some τ_i-open set of $\mathcal{U}$.

Now, we define a weaker concept of a pairwise locally finite open cover than that of Datta in [15, Definition 2.4].

Definition 2.3. A pairwise open cover C of a bitopological space (X,τ_1,τ_2) is said to be *pairwise locally finite* if for each x ∈ X, either there exists a τ_1-open neighborhood of x which meets only a finite number of τ_2-open elements of C, or there exists a τ_2-open neighborhood of x which meets only a finite number of τ_1-open elements of C. The space (X,τ_1,τ_2) is said to be *pairwise paracompact* if every pairwise open cover of X has a locally finite parallel refinement.

Remark 2.1. If $\tau_1=\tau_2$, then the notions of pairwise locally finiteness and pairwise paracompactness coincide with the well known notions of locally finiteness and paracompactness.

We now give the main result of this paper.

Theorem. Every quasi-metric space (X, τ_1, τ_2) is pairwise paracompact.

Proof. Assume d is a compatible quasi-metric on (X, τ_1, τ_2). Let $\mathcal{C} = \{C_i|i\in I\}$ be a pairwise open cover of (X, τ_1, τ_2). By the Well Ordering Principle, there is a well-ordering ≤ on I. Recall that this means that every J ⊂ I has a smallest element with respect to ≤. In particular, for every x, there is a unique i ∈ I, namely $i = \min(\{j \in I|x \in U_j\})$, such that $x \in U_i \ /\cup_{j<i} U_j$. Through induction over n ∈ ℕ, define:

$$V_{i,n} =\cup_{x\in X_{i,n}} B\left(x,\frac{1}{2^n}\right) \text{ if } C_i \text{ is } \tau_1\text{-open} \tag{1}$$

or

$$V_{i,n} =\cup_{x\in X_{i,n}} B^{-1}\left(x,\frac{1}{2^n}\right) \text{ if } C_i \text{ is } \tau_2\text{-open} \tag{2}$$

where

$$X_{i,n} = \{x \in X|B\left(x,\frac{3}{2^n}\right) \subset C_i \text{ if } C_i \text{ is } \tau_1\text{-open} \tag{3}$$

or

$$B^{-1}\left(x,\frac{3}{2^n}\right) \subset C_i \text{ if } C_i \text{ is } \tau_2\text{-open} \tag{4}$$

$$x \notin \cup_{j<i} C_j \cup_{j\in I,k<n} V_{j,k}\}. \tag{5}$$

We show that $\mathcal{V} = \{V_{i,n}|i \in I, n \in \mathbb{N}\}$ is a pairwise locally finite parallel refinement of $\mathcal{C}$, proving pairwise paracompactness.

For fixed i ∈ I and n ∈ ℕ, the set $V_{i,n}$ is either τ_1-open or τ_2-open. Therefore, $B\left(x,\frac{1}{2^n}\right) \subset B\left(x,\frac{3}{2^n}\right) \subset C_i$ or $B^{-1}\left(x,\frac{1}{2^n}\right) \subset B^{-1}\left(x,\frac{3}{2^n}\right) \subset C_i$ for every ball contributing $V_{i,n}$. It follows that $V_{i,n} \subset C_i$. For any x ∈X, taking the smallest i ∈ I with $x \in C_i$. Then, there exists some n ∈ ℕ such that $B\left(x,\frac{3}{2^n}\right) \subset C_i$ or $B\left(x,\frac{3}{2^n}\right) \subset C_i$ depending on whether C_i is τ_1-open or τ_2-open. Then by (3), (4) and (5), since $x \notin \cup_{j<i} C_j$, either $x \in V_{j,k}$ for some j ∈ I and k<n or we have $x \in X_{i,n} \subset V_{i,n}$. Thus $\mathcal{V}$ is a parallel refinement of $\mathcal{C}$ which covers X. To prove that $\mathcal{V}$ is also pairwise locally finite, let x ∈ X.

Define $i = \min\{j \in I|x \in\cup_{n\in\mathbb{N}} V_{j,n}\}$ and suppose that C_i is τ_1-open. Then, we can choose n_0, k ∈ ℕ such that

$$B\left(x,\frac{1}{2^k}\right) \subset V_{i,n_0} \tag{6}$$

We will prove:

(α) If $m \geq n_0 + k$, then for each $V_{j,m} \in \tau_2$, $B(x, \frac{1}{2^{n_0+k}}) \cap V_{j,m} = \emptyset$.

(β) If $m < n_0 + k$, then $B(x, \frac{1}{2^{n_0+k}}) \cap V_{j,m} \neq \emptyset$ for at most $n_0 + k - 1$ $V_{j,m} \in \tau_2$.

To prove (α), let $y \in X_{j,m}$ for some $m \geq n_0 + k$, $j \in I$. Then, in view of $m \geq n_0 + k > n_0$, (5) implies that $y \notin V_{i,n_0}$. Together with $B(x, \frac{1}{2^k}) \subset V_{i,n_0}$, this implies $d(x, y) \geq \frac{1}{2^k}$ ensuring that $B(x, \frac{1}{2^{n_0+k}}) \cap B^{-1}\left(y, \frac{1}{2^m}\right) = \emptyset$. Indeed, suppose to the contrary that $z \in B(x, \frac{1}{2^{n_0+k}}) \cap B^{-1}(y, \frac{1}{2^m})$ for some $z \in X$. Then,

$$d(x,y) \leq d(x,z) + d^{-1}(y,z) < \frac{1}{2^{n_0+k}} + \frac{1}{2^m} < \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k} \tag{7}$$

which is impossible. Consequently, $B(x, \frac{1}{2^{n_0+k}})$ is disjoint from the balls

$$\{B^{-1}\left(y, \frac{1}{2^m}\right) | y \in X_{j,m} \text{ with } j \in I \text{ and } m \geq n_0 + k\} \tag{8}$$

whose union is $V_{j,m}$. The last conclusion proves (α).

To prove (β), let $x \in V_{i,m}$, $y \in V_{j,m}$ where $j \neq i$. By definition of V's, there are $x', y' \in X$ such that $x \in B\left(x', \frac{1}{2^m}\right) \subset V_{i,m}$, $y \in B^{-1}\left(y', \frac{1}{2^m}\right) \subset V_{j,m}$ and $B\left(x', \frac{3}{2^m}\right) \subset C_i$, $B^{-1}\left(y', \frac{3}{2^m}\right) \subset C_j$. Suppose that $i < j$. Then, by $y' \in X_{j,m}$, $i < j$ and (5) we get $y' \notin C_i$. This implies $d(x', y') \geq \frac{3}{2^m}$. Similarly, if $j < i$ then by $x' \in X_{i,m}$, $B^{-1}\left(y', \frac{3}{2^m}\right) \subset C_j$ and (5) we get $x' \notin C_j$. This implies that $d^{-1}(y', x') \geq \frac{3}{2^m}$ or equivalently $d(x', y') \geq \frac{3}{2^m}$.

Therefore, for each $j \in I$ satisfying $j \neq i$ we have

$$d(x', y') \geq \frac{3}{2^m} \tag{9}$$

Thus with the triangle inequality, we have

$$\frac{3}{2^m} \leq d(x', y') \leq d(x', x) + d(x, y) + d^{-1}(y', y) < d(x, y) + \frac{1}{2^m} + \frac{1}{2^m} \tag{10}$$

It follows that

$$d(x,y) \geq \frac{3}{2^m} - \frac{1}{2^m} - \frac{1}{2^m} \geq \frac{1}{2^m} \geq \frac{1}{2^{n_0+k}} \tag{11}$$

This obviously implies

$$B\left(x, \frac{1}{2^{n_0+k}}\right) \cap V_{j,m} = \emptyset \text{ whenever } j \neq i \text{ and } m < n_0 + k \tag{12}$$

Therefore, $B(x, \frac{1}{2^{n_0+k}})$ can meet at most finitely many members of $\mathcal{V}$, namely $V_{i,1}, V_{i,2}, \ldots, V_{i,n_0+k-1}$. This is a proof of the assertion (β).

By supposing that C_i is τ_1-open, it follows that $B(x, \frac{1}{2^{n_0+k}})$ can meet at most $n_0 + k - 1$ τ_2-open elements of $\mathcal{V}$. Similarly, if we suppose that C_i is τ_2-open, then interchanging the roles of τ_1 and τ_2, we see that $B^{-1}(x, \frac{1}{2^{n_0+k}})$ can meet at most

$n_0 + k - 1$ τ_1-open elements of $\mathcal{V}$. Therefore, $\mathcal{V}$ is pairwise locally finite.

If, in the previous theorem, we take $\tau_1 = \tau_2$, we get as corollary the classical result of Stone that metrizable spaces are paracompact.

Corollary 2.1. ([10]) Every metrizable topological space is paracompact.

Rudin [16] has given another proof of the Stone Theorem on paracompactness, which also applies to pseudometrizable topological spaces. The above proof of Theorem 1 still applies to (X, d) when d is only a quasi-pseudometric, since axiom (iii) of definition of quasi-metric is used nowhere.

As mentioned in the introduction, inter alia, pairwise paracompactness is strongly related with locally finite topological spaces, which play an important role in digital topology. The main purpose of digital topology is to study the topological properties of digital images. A digital topological space must obviously be locally finite space to be explicitly representable in the computer (see [17, 18]). Digital images are discrete objects in nature, but usually represent continuous objects or, at least, they are perceived as continuous objects by observers (see [19]). Among locally finite topological spaces, abstract cell complexes are those especially well suited for computer applications. Abstract cell complexes are introduced by Kovalevsky [20] as a means of solving certain connectivity paradoxes in graph-theoretic digital topology, and they can provide an improved theoretical basis for image analysis, which can be illustrated with two examples:

(1) Consider a directed graph (X, V). Then, we can equally characterize (X,V) as a one-dimensional abstract cell complex, where the elements of X are vertices, and the elements of V={(x, y)|x, y $\in$ X} are directed edges from vertex x to y. If we define the topologies τ_1 and τ_2 which have the families {{x}|(x, y) $\in$ V} and {{x}|(y, x) $\in$ V} as bases, respectively, then our graph is a locally finite space since {x} $\in \tau_1 \cap \tau_2$ and |{x}| = 1.

(2) Let ($\mathbb{Z}$, τ,τ^*) be a bitopological space, where for every n $\in \mathbb{Z}$ the family G_n={{a, ..., n, ..., b}|a $\leq$ n $\leq$ b, a, b $\in \mathbb{N}$ ={{n}, {n − 1, n}, {n, n + 1}, {n − 1, n, n + 1}, ...} constitutes a neighborhood system of n for τ , and τ^* is the discrete topology in $\mathbb{N}$. The set G_n is infinite for every n. Additionally, $\mathbb{Z} \in \tau$. Since the singleton set {n} $\in \tau \cap \tau^*$ and |{n}| = 1, we conclude that ($\mathbb{Z}$, τ, τ^*) is locally finite. Clearly, in these two examples the bitopological spaces are pairwise paracompact.

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