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# Accurate Positions of Branch Points of Minimum Magnitudes and Their Associated Spheroidal Eigenvalues

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**Abstract-**The Newton- Raphson method with two complex variable is utilized to compute the branch points with minimum magnitudes and their associated eigenvalues in the first quadrant of the complex plane with the parameter  $c = kF$ , where  $k$  is a complex wave number, and  $F$  is the semifocal length of the spheroidal system. The efficient numerical method which is applied to the spheroidal eigenvalue equation and the equation of its partial derivative with respect to the eigenvalue, is used to simultaneously solve the two complex variables, the branch point and its associated eigenvalue with a high precision, from which they can be tabulated for references.

**Keywords-** Branch Points; Spheroidal Eigenvalues; Newton-Raphson's Method

## I. INTRODUCTION

In this paper, the positions  $c_b$  of the branch points of minimum magnitudes in the complex plane will be accurately calculated with  $c=kF$ , where  $k$  is the operating complex valued wavenumber and  $F$  is the spheroidal semifocal length, and their associated eigenvalues were also calculated. It follows that the magnitudes of these branch points are the convergence radii  $|c_b|$  of the spheroidal power series, as recently published in [1].

Research on the branch points has been done by many authors [1-8]. Meixner, Schafke and Wolf [2] may have been the first persons to calculate the convergence radius of the power series of a spheroidal angle function by using a spheroidal branch point of minimum magnitude. Hunter and Guerrieri [3] classified the spheroidal branch points by utilizing the WKJB method. Barrowes, O'Neill, Grzegorzczuk and Kong [4] refined the work of previous authors and computed the branch points, as well as their associated eigenvalues with high accuracy. Oguchi [5] compared his work with that of Meixner [2] for complex values of propagation constants. The asymptotic iteration for the spheroidal angle function has been used recently to calculate the complex eigenvalues for given complex parameter  $c$  [6]. Skokhodov and D. Khristoforov [7] computed the branch points and their corresponding eigenvalues with high accuracy by using Pade and, Hermite approximants and the generalized Newton iterative method.

For practical reasons, this paper's scope is restricted to the accurate computation of the branch points with minimum magnitudes and their associated eigenvalues since these important spheroidal branch points have some physical meaning; they indicate the convergence radius of a spheroidal power series of a spheroidal angle function of the first kind. Other branch points may not be so interesting or of theoretical interest. For that reason, the author focuses on developing an efficient and accurate algorithm to calculate those branch points that have minimum magnitudes. Moreover, the Newton-Raphson method is chosen even though it is difficult to implement it usually provides a quadratic convergence if the initial trial values are suitably chosen. Moreover, this algorithm relies on the knowledge of the convergence radii whose values are known accurately [1]. Then, the search of these branch points is only in the one dimensional direction (the angular direction in the first quadrant of the complex plane  $c$ ). Then, compared with previous methods of other authors mentioned above, this efficient method should save a lot of computer time, e.g. CPU time less than 0.12 sec for each spheroidal mode. In addition, the tables of these important branch points are not available in the literature. They will be tabulated easily with this fast but accurate algorithm.

The branch points lie between the asymptotic region where the parameter  $c$  is reasonably large and the region where  $c$  is near to zero so that it is difficult to obtain a simple formula for them. Therefore, we have to use numerical methods to calculate them. In the present efficient algorithm to compute these branch points and their associated eigenvalues the complex variables  $c$  and its complex eigenvalue  $\lambda$  are solved simultaneously by using the Newton-Raphson method for 2 complex variables cast in matrix form in which both the spheroidal eigenvalue equation and the condition  $\frac{\partial c^2}{\partial \lambda} = 0$  are satisfied. The initial guess of the complex eigenvalue  $\lambda$  and the branch point  $c$  is facilitated from the power series given in [1] with about 20 terms and from the approximate formulas of the convergence radii  $|c_b|$  derived by the author [1]. As long as the equations are exact the algorithm should provide any required accuracy. In the present time the accuracy can be reached 19 digits for some spheroidal modes with the least amount of CPU time.

II. THE HOMOGENEOUS DIFFERENCE AND TRANSCENDENTAL EQUATIONS

The spheroidal angle function of the first kind,  $S_{mn}(c, \eta)$ , satisfies the following differential equation [1, 9-15]:

$$\frac{d}{d\eta} \left[ (1 - \eta^2) \frac{d}{d\eta} S_{mn}(c, \eta) \right] + \left[ \lambda_{mn} - c^2 \eta^2 - \frac{m^2}{1 - \eta^2} \right] S_{mn}(c, \eta) = 0, m = 0, 1, 2, \dots, n \geq m, \tag{1}$$

where  $c$  is a complex number, and  $\eta$  is the angular coordinate in the prolate spheroidal system. This yields the complex valued eigenvalue  $\lambda_{mn}$ .  $c^2$  can be taken as a point of the entire complex plane. However, in this paper for simplicity  $c^2$  is a point of the upper half complex plane. Then,  $c$  associated with  $\lambda_{mn}$  can be restricted to the first quadrant of the complex plane. It is proved from [1] that  $\lambda^*$  is the eigenvalue associated with  $c^*$ . Therefore, the consideration of the first quadrant is sufficient to obtain all the eigenvalues associated with the complex plane.

We consider the solution  $S_{mn}(c, \eta)$  of (1) in the following expansion in terms of the associated Legendre functions of the first kind [6]:

$$S_{mn}(c, \eta) = \sum_{s=0,1}^{\infty} d_s^{mn}(c) P_{m+s}^m(\eta), \quad m \geq 0; n \geq m, \tag{2}$$

where the index  $s$  runs through even (odd) non-negative integers if  $n-m$  is even (odd). If (2) is substituted into (1) and the recursion relations are used among the associated Legendre functions, after some algebraic manipulation the following homogeneous difference equation emerges [1]:

$$\rho_r c^2 d_{n-m+r+2}^{mn} + (\delta_r + \zeta_r c^2 - \lambda_{mn}) d_{n-m+r}^{mn} + \sigma_r c^2 d_{n-m-r-2}^{mn} = 0, \tag{3}$$

where in [1]

$$\rho_r = \frac{(m+n+r+2)(m+n+r+1)}{(2n+2r+3)(2n+2r+5)}, \tag{4}$$

$$\sigma_r = \frac{(n-m+r)(n-m+r-1)}{(2n+2r-3)(2n+2r-1)}, \tag{5}$$

$$\delta_r = (n+r)(n+r+1), \tag{6}$$

$$\zeta_r = \frac{2(n+r)(n+r+1) - 2m^2 - 1}{(2n+2r-1)(2n+2r+3)}. \tag{7}$$

The index  $r$  in (3) runs through  $l, l+1, l+2, \dots$  with  $l$  being given by [1]

$$l = \begin{cases} -(n-m), & \text{for } (n-m) \text{ even} \\ -(n-m) + 1, & \text{for } (n-m) \text{ odd} \end{cases} \tag{8}$$

Eq. (3) is not suitable for efficient numerical computation. To save computer memories and for efficiency, let

$$r = 2k + l, \tag{9}$$

$$D_k = d_{n-m+r}^{mn} = \hat{d}_r, \tag{10}$$

$$\lambda = \lambda_{mn} \tag{11}$$

Note that  $\hat{d}_r$  was also defined in [1]. The homogeneous difference Eq. (3) can then be represented in a compact and economized form as follows:

$$\hat{\rho}_k c^2 D_{k+1} + (\hat{\delta}_k + \hat{\zeta}_k c^2 - \lambda) D_k + \hat{\sigma}_k c^2 D_{k-1} = 0, \quad k = 0, 1, 2, \dots \tag{12}$$

where

$$\hat{\rho}_k = \rho_{2k+l} = \frac{(m+n+l+2k+2)(m+n+l+2k+1)}{(2n+2l+4k+3)(2n+2l+4k+5)} \tag{13}$$

$$\hat{\sigma}_k = \sigma_{2k+l} = \frac{(n-m+l+2k)(n-m+l+2k-1)}{(2n+2l+4k-3)(2n+2l+4k-1)} \tag{14}$$

$$\hat{\delta}_k = \delta_{2k+l} = (n+l+2k)(n+l+2k+1) \tag{15}$$

$$\hat{\zeta}_k = \zeta_{2k+l} = \frac{2(n+l+2k)(n+l+2k+1) - 2m^2 - 1}{(2n+2l+4k-1)(2n+2l+4k+3)} \tag{16}$$

By letting

$$n_k = \frac{D_k}{D_{k-1}} \tag{17}$$

and by dividing (12) by  $D_k$

$$\hat{\rho}_k c^2 n_{k+1} + \hat{\delta}_k + \hat{\zeta}_k c^2 - \lambda + \frac{\hat{\sigma}_k}{c^2 n_k} = 0, \tag{18}$$

is obtained, from which

$$n_k = -\frac{\hat{\delta}_{k-1} + \hat{\zeta}_{k-1} c^2 - \lambda}{\hat{\rho}_{k-1} c^2} - \frac{\hat{\sigma}_{k-1}}{\hat{\rho}_{k-1} n_{k-1}}, \quad k = 2, 3, \dots \tag{19}$$

This is called a forward recursion relation with the initial value  $n_1$  calculated from (12) with  $k=0$  [6],

$$\hat{\rho}_0 c^2 D_1 + (\hat{\delta}_0 + \hat{\zeta}_0 c^2 - \lambda) D_0 = 0, \tag{20}$$

Yielding

$$n_1 = \frac{D_1}{D_0} = -\frac{\hat{\delta}_0 + \hat{\zeta}_0 c^2 - \lambda}{\hat{\rho}_0 c^2}. \tag{21}$$

The reverse recursion relation is calculated from (18), yielding [6]

$$n_k = -\frac{\hat{\sigma}_k c^2}{\hat{\rho}_k c^2 n_{k+1} + \hat{\delta}_k + \hat{\zeta}_k c^2 - \lambda}. \tag{22}$$

Let  $k = \bar{k} + 1$  be a matching point, a point at which  $n_{\bar{k}+1}$  can be calculated from (19) and (22). Then, by using (19) and (22), a transcendental equation solving for the eigenvalue  $\lambda$  is obtained as follows:

$$\begin{aligned} 0 &= G(c^2, \lambda) = (n_{\bar{k}+1} - n_{\bar{k}}) \hat{\rho}_{\bar{k}} c^2 \\ &= \lambda - \hat{\delta}_{\bar{k}} - \hat{\zeta}_{\bar{k}} c^2 - \frac{\hat{\sigma}_{\bar{k}} c^2}{n_{\bar{k}}} + \frac{\hat{\rho}_{\bar{k}} \hat{\sigma}_{\bar{k}+1} c^4}{\hat{\rho}_{\bar{k}+1} c^2 n_{\bar{k}+2} + \hat{\delta}_{\bar{k}+1} + \hat{\zeta}_{\bar{k}+1} c^2 - \lambda} \end{aligned} \tag{23}$$

where  $n_{\bar{k}}$  is calculated from the forward recursion (19), and  $n_{\bar{k}+2}$  is calculated from the reverse recursion relation (22) assuming that  $\lim_{k \rightarrow \infty} n_k = 0$ .

Taking the partial derivative of (23) with respect to  $\lambda$  gives

$$\frac{\partial G(c^2, \lambda)}{\partial \lambda} = \frac{\partial G(c^2, \lambda)}{\partial c^2} \frac{\partial c^2}{\partial \lambda} = 0 \tag{24}$$

since  $\partial c^2 / \partial \lambda = 0$  ([3]) is the condition under which  $c$  is a spheroidal branch point.

Eqs. (23) and (24) can numerically and simultaneously be solved for a branch point and its associated eigenvalue  $\lambda$  for given for given spheroidal mode  $(m, n)$ .

In the numerical spheroidal computation, let

$$\bar{k} = \frac{2-l}{2} = \begin{cases} \frac{n-m}{2} + 1, & \text{for } (n-m) \text{ even} \\ \frac{n-m+1}{2} & \text{for } (n-m) \text{ odd} \end{cases} \tag{25}$$

To calculate  $n_{\bar{k}+2}$  from the reverse recursion relation,  $N$  must be set such that  $n_N \cong 0$  for  $N = \bar{k} + U \gg 1$ . This may cause some small error in the computation of  $n_{\bar{k}+2}$  if  $N$  is a finite number. Therefore, an approximate expression is developed for  $n_N$  for  $N \gg 1$ .

As shown in [1],

$$\begin{aligned} \hat{d}_r &= \sum_{s=r/2}^{\infty} \hat{d}_{rs} c^{2s} \\ &= D_k, \quad r = 2k + l \end{aligned} \tag{26}$$

Similarly [1],

$$\begin{aligned} \hat{d}_{r-2} &= \sum_{(r-2)/2}^{\infty} \hat{d}_{r-2,s} c^{2s} \\ &= D_{k-1}, \quad r = 2k + l \end{aligned} \tag{27}$$

It follows that

$$\begin{aligned} n_k &= \frac{D_k}{D_{k-1}} \\ &= c^2 \frac{\bar{D}_k}{\bar{D}_{k-1}} \end{aligned} \tag{28}$$

where

$$\bar{D}_k = \sum_{i=0}^{\infty} \bar{d}_{r, \frac{r}{2}+i} c^{2i}, \quad r = 2k + i \tag{29}$$

For  $k = N$ , (28) yields

$$n_N = c^2 \frac{\sum_{i=0}^{\infty} E \bar{d}_{r, \frac{r}{2}+i} c^{2i}}{\sum_{i=0}^{\infty} \bar{d}_{r-2, \frac{r-2}{2}+i} c^{2i}}, \quad r = 2N + l \tag{30}$$

If one leading term is taken in (30),

$$n_N \approx c^2 \frac{\bar{d}_{2s,s}}{\bar{d}_{2(s-1),s-1}}, \quad s = N + l/2 \tag{31}$$

However, Eq. (36) of reference [1], the above ratio can be expressed in a closed form, yielding

$$\begin{aligned} n_N &\approx c^2 \frac{-\sigma_{2s}}{2s(2n + 2s + 1)}, \quad s = N + l/2 \\ &= - \frac{(n-m+l+2N)(n-m+l+2N-1)}{(l+2N)(2n+l+2N+1)(2n+2l+4N-3)(2n+2l+4N-1)} c^2 \end{aligned} \tag{32}$$

The use of expression (32) or (30) can increase the accuracy of the transcendental equation  $G(c^2, \lambda) = 0$ . Note that for large  $N$ ,  $n_N$  varies as

$$n_N \approx - \frac{1}{16} \frac{c^2}{N^2} \tag{33}$$

This shows that as  $N \rightarrow \infty$ ,  $n_N \rightarrow 0$ . In the spheroidal computation,  $U=25$  is used and  $n_N$  is given by (30) with one or two terms in its numerator and denominator to ensure the accuracy of the recursive computation of  $n_{\bar{k}+2}$ .

### III. NEWTON-RAPHSON'S METHOD FOR SPHEROIDAL BRANCH POINTS AND THEIR EIGENVALUES

Let  $x_a = c_a^2$  and  $\lambda = \lambda_a$  be approximate solutions to the transcendental Eqs. (23) and (24). Then the increments  $\Delta x = \Delta c^2$  and  $\Delta \lambda$  are chosen to satisfy

$$0 = G(x_a + \Delta x, \lambda_a + \Delta \lambda) = G(x_a, \lambda_a) + G_x \Delta x + G_\lambda \Delta \lambda \tag{34}$$

$$0 = \frac{\partial G(x_a + \Delta x, \lambda_a + \Delta \lambda)}{\partial \lambda} = G_\lambda(x_a, \lambda_a) + G_{\lambda x} \Delta x + G_{\lambda \lambda} \Delta \lambda \tag{35}$$

where  $G_x = \partial G / \partial x$ ,  $G_\lambda = \partial G / \partial \lambda$ ,  $G_{\lambda x} = \partial^2 G / \partial x \partial \lambda$ ,  $G_{\lambda \lambda} = \partial^2 G / \partial \lambda^2$  are partial derivatives of  $G(x, \lambda)$  evaluated at  $(x_a, \lambda_a)$ .

Eqs. (34) and (35) can be cast in matrix form:

$$\begin{pmatrix} G_x & G_\lambda \\ G_{\lambda x} & G_{\lambda \lambda} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta \lambda \end{pmatrix} = - \begin{pmatrix} G \\ G_\lambda \end{pmatrix} \tag{36}$$

The matrix equation can be solved yielding the following solutions:

$$\Delta x = \frac{G_\lambda^2 - G G_{\lambda \lambda}}{G_x G_{\lambda \lambda} - G_\lambda G_{\lambda x}} \tag{37}$$

$$\Delta \lambda = \frac{G G_{\lambda x} - G_x G_\lambda}{G_x G_{\lambda \lambda} - G_\lambda G_{\lambda x}} \tag{38}$$

Therefore, the improved values of  $x$  and  $\lambda$  are given by

$$x = x_a + \Delta x, \tag{39}$$

$$\lambda = \lambda_a + \Delta \lambda. \tag{40}$$

By computing successively (39) and (40) with  $\Delta x$  and  $\Delta \lambda$  given by (37) and (38)  $x$  and  $\lambda$  approach their true value.

### IV. CALCULATION OF $G$ , $G_x$ , $G_\lambda$ , $G_{\lambda x}$ AND $G_{\lambda \lambda}$

It is proved from (23) that the transcendental equation can be decomposed as follows:

$$0 = G(x, \lambda) = A + \frac{P}{Q} + \frac{R}{S}, \tag{41}$$

where

$$A(x, \lambda) = \lambda - \hat{\delta}_{\bar{k}} - \hat{\zeta}_{\bar{k}} x \tag{42}$$

$$P(x) = \hat{\rho}_{\bar{k}} \hat{\sigma}_{\bar{k}+1} x^2 \tag{43}$$

$$Q(x, \lambda) = \hat{\rho}_{\bar{k}+1}x\hat{n}_{\bar{k}+2} + \hat{\delta}_{\bar{k}+1} + \hat{\zeta}_{\bar{k}+1}x - \lambda \tag{44}$$

$$R(x) = -\hat{\sigma}_{\bar{k}}x \tag{45}$$

$$S(x, \lambda) = n_{\bar{k}}. \tag{46}$$

Then, the second transcendental equation is given by

$$0 = G_\lambda(x, \lambda) = A_\lambda - \frac{PQ_\lambda}{Q^2} - \frac{RS_\lambda}{S^2} \tag{47}$$

where

$$A_\lambda = \frac{\partial A}{\partial \lambda} = 1 \tag{48}$$

$$Q_\lambda = \frac{\partial Q}{\partial \lambda} = \hat{\rho}_{\bar{k}+1}x n_{\bar{k}+2, \lambda} - 1 \tag{49}$$

$$S_\lambda = \frac{\partial S}{\partial \lambda} = n_{\bar{k}, \lambda} \tag{50}$$

It is straightforward to show that

$$G_x = A_x + \frac{P_x Q - P Q_x}{Q^2} + \frac{R_x S - R S_x}{S^2} \tag{51}$$

where

$$A_x = \frac{\partial A}{\partial x} = -\hat{\zeta}_{\bar{k}} \tag{52}$$

$$P_x = \frac{\partial P}{\partial x} = 2\hat{\rho}_{\bar{k}}\hat{\sigma}_{\bar{k}+1}x \tag{53}$$

$$Q_x = \frac{\partial Q}{\partial x} = \hat{\rho}_{\bar{k}+1}n_{\bar{k}+2} + \hat{\rho}_{\bar{k}+1}x n_{\bar{k}+2, x} + \hat{\zeta}_{\bar{k}+1} \tag{54}$$

$$R_x = \frac{\partial R}{\partial x} = -\hat{\sigma}_{\bar{k}} \tag{55}$$

$$S_x = \frac{\partial S}{\partial x} = n_{\bar{k}, x} \tag{56}$$

Taking the partial derivative of (47) with respect to  $x$  and  $\lambda$  gives

$$G_{\lambda\lambda} = \frac{\partial^2 G}{\partial \lambda^2} = -\frac{PQQ_{\lambda\lambda} - 2PQ_\lambda^2}{Q^3} - \frac{RSS_{\lambda\lambda} - 2RS_\lambda^2}{S^3} \tag{57}$$

$$G_{\lambda x} = \frac{\partial^2 G}{\partial x \partial \lambda} = \frac{(P_x Q_\lambda + P Q_{\lambda x})Q - 2P P_x Q_\lambda}{Q^3} - \frac{(R_x S_\lambda + R S_{\lambda x})S - 2R R_x S_\lambda}{S^3} \tag{58}$$

where

$$Q_{\lambda\lambda} = \frac{\partial^2 Q}{\partial \lambda^2} = \hat{\rho}_{\bar{k}+1}x n_{\bar{k}+2, \lambda\lambda} \tag{59}$$

$$S_{\lambda\lambda} = \frac{\partial^2 S}{\partial \lambda^2} = n_{\bar{k}, \lambda\lambda} \tag{60}$$

$$Q_{\lambda x} = \frac{\partial^2 Q}{\partial x \partial \lambda} = \hat{\rho}_{\bar{k}+1}n_{\bar{k}+2, \lambda} + \hat{\rho}_{\bar{k}+1}x n_{\bar{k}+2, \lambda x} \tag{61}$$

$$S_{\lambda x} = \frac{\partial^2 S}{\partial x \partial \lambda} = n_{\bar{k}, \lambda x} \tag{62}$$

To compute  $G(x, \lambda)$  as well as its partial derivatives as required by (37), (38) in the Raphson-Newton method, as shown in this section the forward recursion relations and those for their partial derivatives are needed to calculate them at  $k = \bar{k}$ . Similarly, the reverse recursion relations and those for their partial derivatives are also required to calculate them at  $k = \bar{k} + 2$ .

The expressions of the forward and reverse recursion relations for  $n_k, n_{k,x}, n_{k,\lambda}, n_{k,\lambda\lambda}$  and  $n_{k,\lambda x}$  are derived, and all of them all shown in Appendix A.

#### V. ALGORITHM AND NUMERICAL RESULTS FOR COMPUTATION OF BRANCH POINTS WITH MINIMUM MAGNITUDES AND THEIR ASSOCIATED EIGENVALUES

To initiate the algorithm of computation for these branch points of a given spheroidal mode  $(m, n)$ , it is crucial to have the initial values of a branch point  $c=c_0$  and its associated eigenvalue  $\lambda = \lambda_0$  since the Newton-Raphson method solves the complex variables  $c^2 (=x)$  and  $\lambda$  simultaneously. Let  $c_0$  be restricted such that  $R_{\min} \leq |c_0| \leq R_{\max} < R$ , where  $R$  is a radius of convergence

of a given spheroidal mode  $(m, n)$ . As shown in the author's recent paper [1] the convergence radius  $R$  is known accurately so that this information can be used to numerically compute the approximate branch points and the associated eigenvalues.

$\Delta_R$  is also defined by

$$\Delta_R = \frac{R_{max} - R_{min}}{N_R} \quad (63)$$

where  $N_R$  is the number of radial divisions in the radial direction in the first quadrant of the complex plane. In the practical computation  $N_R = 1$ . For  $N_R = 1$ ,  $N_{max} = R - e$  and  $R_{min} = R - 2e$ , were chosen, where  $e > 0$  is a small number.

$\Delta_\theta$  is also defined by

$$\Delta_\theta = \frac{\theta_{max} - \theta_{min}}{N_\theta}, \quad \theta_{max} = \frac{\pi}{2}, \quad \theta_{min} = 0, \quad (64)$$

where  $N_\theta$  is the number of angular divisions in the angular direction from 0 to  $\pi/2$ . In the computation  $N_\theta = 30$ .

The sampled points in the radial direction are

$$R_r = R_{min} + (r - 1)\Delta_R, \quad r = 1, 2, \dots, N_R + 1. \quad (65)$$

The sampled points in the angular direction are

$$\theta_s = \theta_{min} + \frac{\Delta_\theta}{2} + (s - 1)\Delta_\theta, \quad s = 1, 2, \dots, N_\theta. \quad (66)$$

Therefore, the sampled points  $c = c_{rs}$  are given by

$$\begin{aligned} c_{rs} &= R_r e^{i\theta_s} \\ &= R_r (\cos\theta_s + i\sin\theta_s), \quad r = 1, 2, \dots, N_R + 1; \quad s = 1, 2, \dots, N_\theta \end{aligned} \quad (67)$$

The associated eigenvalues  $\lambda = \lambda_{rs}$  are computed from the power series given by Do-Nhat [Eq. (10), 1],

$$\lambda_{rs} = \sum_{k=0}^{\infty} \Gamma_k c_{rs}^{2k} \quad (68)$$

since  $|c_{rs}|$  is less than the convergence radius  $R$  for a given spheroidal mode. The maximum number of terms in the power series is less than or equal to 21. Note that as shown in [1] the computation of the coefficients  $\Gamma_k$  is very fast since they are related to recursion relations.

The number of iterations was also set in the Newton- Raphson method (NRM). It is less than or equal to 15 due to the fast convergence of the algorithm.

The computer program begins by calculating  $\bar{k}$  given by (25), and  $N = \bar{k} + U = \bar{k} + 25$  for a given spheroidal mode  $(m, n)$ . An  $R_{max}$  is estimated that is near to but less than the convergence radius  $R$ .  $R_{min}$  is around  $R_{max}$ , but  $0 < R_{min} \leq R_{max}$ . Hence, the number of radial divisions  $N_R$  is small. For each  $r$  ( $r=1, 2, \dots, N_R + 1$ )  $c_{rs}$  and  $\lambda$  are calculated according to (67), and (68) as starting solutions to NRM.

For constant number of iterations  $ITE \leq 15$ , the NRM is activated by computing  $x (=c^2)$  and  $\lambda$  from (39) and (40) with  $\Delta x$  and  $\Delta \lambda$  given by (37) and (38).

In the process of NRM activation  $G$ ,  $G_x$ ,  $G_\lambda$ ,  $G_{\lambda x}$  and  $G_{\lambda\lambda}$  are calculated from (41)-(62), with  $n_{\bar{k}}$ ,  $n_{\bar{k},x}$ ,  $n_{\bar{k},\lambda}$ ,  $n_{\bar{k},\lambda\lambda}$ ,  $n_{\bar{k},\lambda x}$  being computed from the forward recursion relations given by (19), (A-1), (A-3), (A-5), (A-7), and with  $n_{\bar{k}+2}$ ,  $n_{\bar{k}+2,x}$ ,  $n_{\bar{k}+2,\lambda}$ ,  $n_{\bar{k}+2,\lambda\lambda}$ ,  $n_{\bar{k}+2,\lambda x}$  being computed from the reverse recursion relations given by (A-18), (A-19), (A20), (A-21) and (A-22).

Here, it is of interest to note that in the reverse process we calculate  $n_N$ ,  $n_{N,x}$  given by (A-9) and (A-14) and  $n_{N,\lambda} = n_{N,\lambda\lambda} = n_{N,\lambda x} = 0$  as by given (A-15), (A-16), and (A-17) in order to obtain much more accuracy with  $N$  as small as possible.

Two terms each were used for  $\bar{D}_N$  and  $\bar{D}_{N-1}$ , whose coefficients are calculated from [1]. Usually, other authors set  $n_N = 0$ . This causes the computation to become less accurate since from (33)  $n_N$  varies as  $-c^2/(16N^2)$ . It follows that the error is large if  $N$  is small.

To implement the condition for minimum magnitudes of the branch points, the branch point was chosen among others at which its magnitude  $|c|$  is minimal since the NRM, in theory, yields a finite set of branch points for a given spheroidal mode  $(m, n)$ . If the NRM finds a branch point of minimum magnitude more than one time for the given spheroidal mode, then the NRM choose the desired branch point with the least error. This is the most accurate branch point determined by the NRM from which the initial position of  $c$  corresponding to  $\lambda$  is located. The error of computing a branch point  $c$  and its associated eigenvalue  $\lambda$  for the NRM is defined by the Euclidean norm,

$$e_3(c) = \|\vec{\Delta}_3(c)\| \quad (69)$$

where

$$\vec{\Delta}_3(c) = (\text{Re}(\sqrt{\Delta x}), \text{Im}(\sqrt{\Delta x}), \text{Re}(\Delta \lambda), \text{Im}(\Delta \lambda)). \tag{70}$$

$\Delta x$  and  $\Delta \lambda$  are the increments of  $x$  and  $\lambda$  at the maximum iteration number ITE. The optimal branch point  $c$  and its optimal associated eigenvalue  $\lambda$  for the NRM are such that

$$e_3(c) = \min_{d \in B} e_3(d), \tag{71}$$

where  $B$  is the set of the values of the same branch points having different errors  $e_3(d)$  in the computation using the NRM. Therefore, (71) is the criterion for the NRM to choose the most accurate branch point having a minimum magnitude and its most accurate associated eigenvalue. Usually  $e_3(c) = 10^{-13}$  yields  $c$  and  $\lambda$  of about 14 digits of accuracy.

Note that other errors of computing  $c$  and  $\lambda$  are also defined for the NRM:

$$e_1(c) = \|\vec{\Delta}_1(c)\| \tag{72}$$

$$e_2(\lambda) = \|\vec{\Delta}_2(\lambda)\| \tag{73}$$

where

$$\Delta_1(c) = (\text{Re}(\sqrt{\Delta x}), \text{Im}(\sqrt{\Delta x})) \tag{74}$$

$$\Delta_2(\lambda) = (\text{Re}(\Delta \lambda), \text{Im}(\Delta \lambda)) \tag{75}$$

The errors  $e_1(c)$  and  $e_2(\lambda)$  given by (72) and (73) are used to check the computing accuracy of the branch points and that of the corresponding eigenvalues.

The computed numerical results using the NRM of these branch points and their corresponding eigenvalues for a variety of spheroidal modes are too numerous to record in this paper. However, recorded data is provided, for reference, such as  $c_b$ , branch points with minimum magnitude in the first quadrant of the complex plane, and their associated eigenvalues  $\lambda$  for some low spheroidal modes  $(m, n)$  with  $m=0, 1, 2, 3$  and with  $n=m, m+1, \dots, 40$ . The data is shown in Tables 1, 2, 3 and 4 for  $m=0, m=1, m=2$  and  $m=3$ , respectively. The branch points and the associated eigenvalues computed by Barrowes, O'Neill, Grzegorzczuk, Kong [4] agree with those using the present NRM with 13 digits for  $m=0, 1, 2, 3$  and  $n=m, \dots, 6, 7, 8$ . They are shown in Tables 1, 2, 3, and 4.

Table 5 compares the computed results of the eigenvalues at the branch points of minimum magnitudes with those of Oguchi [5] and [8]. It seems that the numerical results of Li, Leong, Yeo, Kooi, Tan [8] are less accurate since the current results agree very well with those of [4].

By using the entries of the 3<sup>rd</sup> column of Table 2 the magnitudes of  $c_b$  are calculated using the NRM, yielding the convergence radii of the spheroidal symmetric modes  $m=1$  for  $n=1, 2, \dots, 40$ , shown in Table 6. Some of these convergence radii are compared with those of Meixner, Schafke, Wolf [2] and are also recorded in Table 6. There is a high precision agreement between the compared data sets.

The scilab simulator was used to write the program of the above branch point problem. For each computational point, the branch point value and the associated spheroidal eigenvalue, the CPU time was less than 0.12 sec for  $N_R = 1$  given the known convergence radius  $R$  of a spheroidal mode  $(m, n)$ .

TABLE 1 POSITIONS OF THE BRANCH POINTS FOR  $m = 0$  ON THE CIRCLE OF CONVERGENCE (IN THE FIRST QUADRANT) OF THE SPHEROIDAL EIGENVALUE POWER SERIES IN THE COMPLEX PLANE AND THEIR ASSOCIATED EIGENVALUES COMPUTED IN HIGH PRECISION

$m = 0$	$n$	$c_b$	$\lambda_b$
	0,2	1.824770749209D+00 + 2.601670692890D+00i	1.705180091412D+00 + 4.220186348356D+00i
From [4]	0,2	1.824770749209D+00 + 2.601670692890D+00i	1.705180091412D+00 + 4.220186348356D+00i
	1,3	3.563644553545D+00 + 2.887165344337D+00i	1.014083878723D+01 + 1.112158668422D+01i
From [4]	1,3	3.563644553545D+00 + 2.887165344337D+00i	1.014083878723D+01 + 1.112158668422D+01i
	4	5.217093042405D+00 + 3.081362886558D+00i	2.391582957441D+01 + 1.874332721627D+01i
From [4]	4	5.217093042405D+00 + 3.081362886558D+00i	2.391582957441D+01 + 1.874332721627D+01i
	5	4.067274712533D+00 + 6.264358978588D+00i	1.178092583829D+01 + 2.254139320796D+01i
From [4]	5	4.067274712533D+00 + 6.264358978588D+00i	1.178092583829D+01 + 2.254139320796D+01i
	6	5.874514148329D+00 + 6.617314890038D+00i	2.751644535984D+01 + 3.776498428325D+01i
From [4]	6	5.874514148329D+00 + 6.617314890038D+00i	2.751644535984D+01 + 3.776498428325D+01i
	7	7.606334073445D+00 + 6.906465157219D+00i	4.880665241411D+01 + 5.401198982276D+01i
From [4]	7	7.606334073445D+00 + 6.906465157219D+00i	4.880665241411D+01 + 5.401198982276D+01i
	8	9.295569705184D+00 + 7.151702724613D+00i	7.546220808381D+01 + 7.111265817867D+01i
From [4]	8	9.295569705184D+00 + 7.151702724613D+00i	7.546220808381D+01 + 7.111265817867D+01i
	9	8.147467516662D+00 + 1.032389930138D+01i	5.283139589301D+01 + 7.922201847555D+01i
	10	9.917892969756D+00 + 1.065108985850D+01i	8.146539071472D+01 + 1.038972143495D+02i
	11	1.164803165683D+01 + 1.094178078184D+01i	1.155848949946D+02 + 1.295758724802D+02i
	12	1.334971299760D+01 + 1.120343527735D+01i	1.550810251652D+02 + 1.561483681746D+02i

	13	1.220345228870D+01 + 1.437198974974D+01i	1.220428033294D+02 + 1.685490785182D+02i
	14	1.396027395268D+01 + 1.468902789400D+01i	1.635280820125D+02 + 2.026831514670D+02i
	15	1.568963967410D+01 + 1.498047180460D+01i	2.104727027093D+02 + 2.378112379834D+02i
	16	1.739763177801D+01 + 1.525021911045D+01i	2.627998771492D+02 + 2.738525159641D+02i
	17	1.625258010907D+01 + 1.841692129135D+01i	2.193790959653D+02 + 2.905374619865D+02i
	18	1.800231661175D+01 + 1.872851957742D+01i	2.737018299674D+02 + 3.341329385353D+02i
	19	1.973123632897D+01 + 1.902040559384D+01i	3.334701354988D+02 + 3.787169813001D+02i
	20	2.144304160560D+01 + 1.929497145653D+01i	3.986246268167D+02 + 4.242256014731D+02i
	21	2.029881477019D+01 + 2.246054814130D+01i	3.448317025570D+02 + 6.95034083201D+02i
	22	2.204420475145D+01 + 2.276875453828D+01i	4.119859627662D+02 + 4.982495281163D+02i
	23	2.377283382228D+01 + 2.306093061880D+01i	4.845772964464D+02 + 5.522926324119D+02i
	24	2.548719199886D+01 + 2.333869339952D+01i	5.625572651765D+02 + 6.072678123285D+02i
	25	2.434355818578D+01 + 2.650351279702D+01i	4.983975404026D+02 + 6.325127522485D+02i
	26	2.60860095695D+01 + 2.68094015931D+01i	5.78380241415D+02 + 6.95034083201D+02i
	27	2.78144341128D+01 + 2.71017824191D+01i	6.63794248253D+02 + 7.58537966071D+02i
	28	2.95306209603D+01 + 2.73818211578D+01i	7.54598652269D+02 + 8.22979235762D+02i
	29	2.83874324092D+01 + 3.05460958701D+01i	6.80075229325D+02 + 8.52502161436D+02i
	30	3.01277643805D+01 + 3.08503006223D+01i	7.72884561277D+02 + 9.24487152451D+02i
	31	3.18560371406D+01 + 3.11428336221D+01i	8.71121027880D+02 + 9.974528622509D+02i
	32	3.357359828D+01 + 3.142457521D+01i	9.747492198D+02 + 1.071359918D+03i
	33	3.243075573D+01 + 3.458843920D+01i	8.898640591D+02 + 1.105160113D+03i
	34	3.416948684D+01 + 3.489136487D+01i	9.954988697D+02 + 1.186609028D+03i
	35	3.589764247D+01 + 3.518401538D+01i	1.106557658D+03 + 1.269037251D+03i
	36	3.761627413D+01 + 3.546707896D+01i	1.223009208D+03 + 1.35240988634D+03i
	37	3.647370795D+01 + 3.863062214D+01i	1.127763627D+03 + 1.390486857D+03i
	38	3.821118718D+01 + 3.893254326D+01i	1.246223138D+03 + 1.481399880D+03i
	39	3.993924965D+01 + 3.922528719D+01i	1.370104153D+03 + 1.573291087D+03i
	40	4.16587387D+01 + 3.95094069D+01i	1.499378761D+03 + 1.666129156D+03i

TABLE 2 POSITIONS OF THE BRANCH POINTS FOR M=1 ON THE CIRCLE OF CONVERGENCE (IN THE FIRST QUADRANT) OF THE SPHEROIDAL EIGENVALUE POWER SERIES IN THE COMPLEX PLANE AND THEIR ASSOCIATED EIGENVALUES COMPUTED IN HIGH PRECISION

$m = 1$	$n$	$c_b$	$\lambda_b$
	1,3	1.998555442182D+00 + 4.097453662365D+00i	2.915318658097D+00 + 6.133951063223D+00i
From [4]	1,3	1.998555442182D+00 + 4.097453662365D+00i	2.915318658097D+00 + 6.133951063223D+00i
	2,4	3.862833529249D+00 + 4.492300074954D+00i	1.220109599352D+01 + 1.624407908634D+01i
From [4]	2,4	3.862833529249D+00 + 4.492300074954D+00i	1.220109599352D+01 + 1.624407908634D+01i
	5	5.594491061916D+00 + 4.781582485576D+00i	2.713407308522D+01 + 2.742376175176D+01i
From [4]	5	5.594491061916D+00 + 4.781582485576D+00i	2.713407308522D+01 + 2.742376175176D+01i
	6	7.270040170458D+00 + 5.010809182556D+00i	4.741098535765D+01 + 3.942617876535D+01i
From [4]	6	7.270040170458D+00 + 5.010809182556D+00i	4.741098535765D+01 + 3.942617876535D+01i
	7	6.119087892219D+00 + 8.234638882859D+00i	2.988062405105D+01 + 4.618127876695D+01i
From [4]	7	6.119087892219D+00 + 8.234638882859D+00i	2.988062405105D+01 + 4.618127876695D+01i
	8	7.901814171762D+00 + 8.572100476598D+00i	5.211255309257D+01 + 6.592101134661D+01i
From [4]	8	7.901814171762D+00 + 8.572100476598D+00i	5.211255309257D+01 + 6.592101134661D+01i
	9	9.632074824342D+00 + 8.862803308220D+00i	7.985065195384D+01 + 8.667817445274D+01i
	10	1.132829969426D+01 + 9.118350208514D+00i	1.129575186973D+02 + 1.083173449853D+02i
	11	1.018056379970D+01 + 1.230619870923D+01i	8.503931806024D+01 + 1.19062935286D+02i
	12	1.194266364540D+01 + 1.262783181629D+01i	1.201250668136D+02 + 1.483260570383D+02i
	13	1.367212407829D+01 + 1.291927329074D+01i	1.606791391225D+02 + 1.785898527018D+02i
	14	1.537719479862D+01 + 1.318580659960D+01i	2.066116875704D+02 + 2.097606185234D+02i
	15	1.423126543442D+01 + 1.636257995717D+01i	1.683235289060D+02 + 2.246589569292D+02i
	16	1.598392516106D+01 + 1.667677726584D+01i	2.162458888235D+02 + 2.634168538438D+02i
	17	1.771291228630D+01 + 1.696865893167D+01i	2.696188292292D+02 + 3.031672095440D+02i
	18	1.942289998260D+01 + 1.724124367529D+01i	3.283754602097D+02 + 3.438383115698D+02i
	19	1.827809453947D+01 + 2.041300176218D+01i	2.797234107249D+02 + 3.629428868380D+02i
	20	2.002533914690D+01 + 2.072287826656D+01i	3.404761391151D+02 + 4.111837911545D+02i
	21	2.175401747820D+01 + 2.101505021303D+01i	4.066691326987D+02 + 4.604122760044D+02i
	22	2.346713635786D+01 + 2.129147004492D+01i	4.782490774513D+02 + 5.105702417593D+02i
	23	2.232309327800D+01 + 2.446046864360D+01i	4.192359797868D+02 + 5.339055911069D+02i
	24	2.406682544055D+01 + 2.476752071121D+01i	4.928161096920D+02 + 5.916235258346D+02i
	25	2.579528725221D+01 + 2.505989797417D+01i	5.718297561375D+02 + 6.503258118440D+02i
	26	2.75105752129D+01 + 2.53389665373D+01i	6.56232586969D+02 + 7.0996611563D+02i
	27	2.63670821780D+01 + 2.85062550386D+01i	5.86860012496D+02 + 7.37543078925D+02i
	28	2.81083514992D+01 + 2.88113163925D+01i	6.73265900914D+02 + 8.04734595558D+02i
	29	2.98366533346D+01 + 2.91038463854D+01i	7.65100545432D+02 + 8.72908168320D+02i
	30	3.15535327678D+01 + 2.93848543479D+01i	8.62326000170D+02 + 9.42021671539D+02i
	31	3.04104549715D+01 + 3.25509943809D+01i	7.82594912017D+02 + 9.73853331732D+02i
	32	3.2149901698D+01 + 3.2854577056D+01i	8.8182555377D+02 + 1.0505162607D+03i
	33	3.3878080604D+01 + 3.3147225036D+01i	9.8648141321D+02 + 1.1281595306D+03i
	34	3.5596177559D+01 + 3.3429714467D+01i	1.0965293197D+03 + 1.2067461045D+03i
	35	3.445342421D+01 + 3.659503721D+01i	1.006440352D+03 + 1.242835219D+03i

	36	3.619146763D+01 + 3.689747819D+01i	1.118495087D+03 + 1.328968107D+03i
	37	3.791954915D+01 + 3.719022004D+01i	1.235972306D+03 + 1.416080006D+03i
	38	3.96386077D+01 + 3.74738781D+01i	1.35884255D+03 + 1.50413787D+03i
	39	3.84961148D+01 + 4.06385933D+01i	1.25839614D+03 + 1.54448806D+03i
	40	4.02330444D+01 + 4.09401261D+01i	1.38327451D+03 + 1.64008989D+03i

TABLE 3 POSITIONS OF THE BRANCH POINTS FOR  $m = 2$  ON THE CIRCLE OF CONVERGENCE (IN THE FIRST QUADRANT) OF THE SPHEROIDAL EIGENVALUE POWER SERIES IN THE COMPLEX PLANE AND THEIR ASSOCIATED EIGENVALUES COMPUTED IN HIGH PRECISION

$m = 2$	$n$	$c_b$	$\lambda_b$
	2,4	2.136987377094D+00 + 5.449457313914D+00i	6.102540356950D+00 + 7.684763813617D+00i
From [4]	2,4	2.136987377094D+00 + 5.449457313914D+00i	6.102540356950D+00 + 7.684763813617D+00i
	3,5	4.105156484216D+00 + 5.922750658440D+00i	1.613686626880D+01 + 2.040623063149D+01i
From [4]	3,5	4.105156484216D+00 + 5.922750658440D+00i	1.613686626880D+01 + 2.040623063149D+01i
	6	5.907125751703D+00 + 6.283464345814D+00i	3.208758614874D+01 + 3.448753467768D+01i
From [4]	6	5.907125751703D+00 + 6.283464345814D+00i	3.208758614874D+01 + 3.448753467768D+01i
	7	7.634658702877D+00 + 6.576768822064D+00i	5.356137110171D+01 + 4.963851594122D+01i
From [4]	7	7.634658702877D+00 + 6.576768822064D+00i	5.356137110171D+01 + 4.963851594122D+01i
	8	9.320494635223D+00 + 6.824432371374D+00i	8.037603943595D+01 + 6.566897041441D+01i
From [4]	8	9.320494635223D+00 + 6.824432371374D+00i	8.037603943595D+01 + 6.566897041441D+01i
	9	8.166812656483D+00 + 1.011835701797D+01i	5.729687751368D+01 + 7.639774651331D+01i
	10	9.935661817903D+00 + 1.044808878159D+01i	8.604814314068D+01 + 1.003886588347D+02i
	11	1.166438456659D+01 + 1.074057147836D+01i	1.202745887053D+02 + 1.253987500874D+02i
	12	1.336481866704D+01 + 1.100353325020D+01i	1.598687078213D+02 + 1.513138513180D+02i
	13	1.221643900406D+01 + 1.422439606386D+01i	1.265490695215D+02 + 1.654168182745D+02i
	14	1.397251063089D+01 + 1.454269393678D+01i	1.681152512438D+02 + 1.990605192164D+02i
	15	1.570117830240D+01 + 1.483512573728D+01i	2.151356587933D+02 + 2.337061178617D+02i
	16	1.740852909501D+01 + 1.510566185202D+01i	2.675339891271D+02 + 2.692710952169D+02i
	17	1.626235670266D+01 + 1.830166959603D+01i	2.239067569612D+02 + 2.872295894580D+02i
	18	1.801165562246D+01 + 1.861402014998D+01i	2.782913107454D+02 + 3.304425866790D+02i
	19	1.974016069775D+01 + 1.890653046917D+01i	3.381183821997D+02 + 3.746488847000D+02i
	20	2.145157658032D+01 + 1.918162022689D+01i	4.033288106452D+02 + 4.197836826459D+02i
	21	2.030665498301D+01 + 2.236597633444D+01i	3.493725598188D+02 + 4.417699013299D+02i
	22	2.205175843549D+01 + 2.267468276850D+01i	4.165768622966D+02 + 4.945143558641D+02i
	23	2.378011304445D+01 + 2.296728869339D+01i	4.89216208897D+02 + 5.482469979413D+02i
	24	2.549421003111D+01 + 2.324542377115D+01i	5.672422835470D+02 + 6.029144225742D+02i
	25	2.43501028892D+01 + 2.64233148678D+01i	5.02947355764D+02 + 6.29011545468D+02i
	26	2.60923521465D+01 + 2.67295601215D+01i	5.82972099388D+02 + 6.91267059906D+02i
	27	2.782058167333D+01 + 2.70222548125D+01i	6.68426702465D+02 + 7.54507381432D+02i
	28	2.95365812952D+01 + 2.73025714196D+01i	7.59270330156D+02 + 8.18687069746D+02i
	29	2.83930493533D+01 + 3.04764727383D+01i	6.84631525806D+02 + 8.48942019481D+02i
	30	3.01332311123D+01 + 3.07809444498D+01i	7.77477109697D+02 + 9.20696331134D+02i
	31	3.1861358195D+01 + 3.1073716643D+01i	8.7574874512D+02 + 9.9343305285D+02i
	32	3.3578778599D+01 + 3.1355673423D+01i	9.7941107076D+02 + 1.0671126315D+03i
	33	3.2435675477D+01 + 3.4526923857D+01i	8.944252632D+02 + 1.101554593D+03i
	34	3.417429057D+01 + 3.483005693D+01i	1.000091939D+03 + 1.182799751D+03i
	35	3.59023332D+01 + 3.51228958D+01i	1.11118175D+03 + 1.26502553D+03i
	36	3.762085528D+01 + 3.540613086D+01i	1.227663516D+03 + 1.348196916D+03i
	37	3.647808456D+01 + 3.857552149D+01i	1.132328676D+03 + 1.386845318D+03i
	38	3.821547148D+01 + 3.887760841D+01i	1.250816613D+03 + 1.477575872D+03i
	39	3.99434438D+01 + 3.91705045D+01i	1.37472538D+03 + 1.56928566D+03i
	40	4.16628451D+01 + 3.94547640D+01i	1.50402710D+03 + 1.66194328D+03i

TABLE 4 POSITIONS OF THE BRANCH POINTS FOR  $m = 3$  ON THE CIRCLE OF CONVERGENCE (IN THE FIRST QUADRANT) OF THE SPHEROIDAL EIGENVALUE POWER SERIES IN THE COMPLEX PLANE C AND THEIR ASSOCIATED EIGENVALUES COMPUTED IN HIGH PRECISION

$m = 3$	$n$	$c_b$	$\lambda_b$
	3,5	2.254441944326D+00 + 6.731940814253D+00i	1.127373902543D+01 + 9.046369132359D+00i
From [4]	3,5	2.254441944326D+00 + 6.731940814253D+00i	1.127373902543D+01 + 9.046369132359D+00i
	4,6	4.312789375877D+00 + 7.267942971679D+00i	2.198994478605D+01 + 2.406425134282D+01i
From [4]	4,6	4.312789375877D+00 + 7.267942971679D+00i	2.198994478605D+01 + 2.406425134282D+01i
	7	6.178462421804D+00 + 7.686866389129D+00i	3.886808748068D+01 + 4.069988568352D+01i
From [4]	7	6.178462421804D+00 + 7.686866389129D+00i	3.886808748068D+01 + 4.069988568352D+01i
	8	7.954733051350D+00 + 8.033437876319D+00i	6.143614311277D+01 + 5.862466869852D+01i
From [4]	8	7.954733051350D+00 + 8.033437876319D+00i	6.143614311277D+01 + 5.862466869852D+01i
	9	9.679537838481D+00 + 8.329895457322D+00i	8.947076001505D+01 + 7.761930390071D+01i
	10	1.137120739361D+01 + 8.589281166145D+00i	1.228421434513D+02 + 9.752961467855D+01i
	11	1.021368756727D+01 + 1.195273090013D+01i	9.408140278588D+01 + 1.128800094094D+02i
	12	1.197349190560D+01 + 1.227800968655D+01i	1.293670815054D+02 + 1.409707544773D+02i
	13	1.370086305467D+01 + 1.257218021544D+01i	1.701058907860D+02 + 1.70084277234D+02i
	14	1.540405708979D+01 + 1.284080777678D+01i	2.162097953029D+02 + 2.001213577543D+02i
	15	1.425438654823D+01 + 1.609957672939D+01i	1.773968738940D+02 + 2.180912565124D+02i
	16	1.600586370461D+01 + 1.641574485425D+01i	2.254627942174D+02 + 2.559771011844D+02i

17	1.773374270486D+01 + 1.670922053442D+01i	2.789711848917D+02 + 2.948675871098D+02i
18	1.944270213756D+01 + 1.698311291787D+01i	3.378558600299D+02 + 3.346886612020D+02i
19	1.829590115099D+01 + 2.020309325505D+01i	2.888172363196D+02 + 3.561314129949D+02i
20	2.004242275367D+01 + 2.051420675231D+01i	3.496822814687D+02 + 4.03676586174D+02i
21	2.177041308947D+01 + 2.080742481131D+01i	4.159825292593D+02 + 4.522170603101D+02i
22	2.348288199202D+01 + 2.108473746128D+01i	4.876650119994D+02 + 5.016935283969D+02i
23	2.233759049002D+01 + 2.428564834151D+01i	4.283443009096D+02 + 5.269255753958D+02i
24	2.408083425102D+01 + 2.459354995411D+01i	5.020167881848D+02 + 5.840642726564D+02i
25	2.58088268777D+01 + 2.48866668468D+01i	5.81119325157D+02 + 6.42192652926D+02i
26	2.75236665707D+01 + 2.51663834784D+01i	6.65607775655D+02 + 7.01261175284D+02i
27	2.63793156286D+01 + 2.83564015800D+01i	5.95979142401D+02 + 7.30439486180D+02i
28	2.812023264607D+01 + 2.866208275218D+01i	6.824635010152D+02 + 7.971346259565D+02i
29	2.98481938535D+01 + 2.89551635171D+01i	7.74374090688D+02 + 8.64815725530D+02i
30	3.15647453490D+01 + 2.92366632153D+01i	8.7167308731D+02 + 9.3344021387D+02i
31	3.0421040300D+01 + 3.2419833943D+01i	7.917224146D+02 + 9.666552108D+02i
32	3.21602207763D+01 + 3.27238889409D+01i	8.910212933D+02 + 1.042883780D+03i
33	3.3888141055D+01 + 3.3016963049D+01i	9.957434675D+02 + 1.120095672D+03i
34	3.560598764D+01 + 3.329983833D+01i	1.1058558476D+03 + 1.1982535502D+03i
35	3.446275502D+01 + 3.647840565D+01i	1.015574532D+03 + 1.235562435D+03i
36	3.620059009D+01 + 3.678121859D+01i	1.1276896447D+03 + 1.3213091500D+03i
37	3.792846852D+01 + 3.707429995D+01i	1.245225729D+03 + 1.408037228D+03i
38	3.964732958D+01 + 3.735826886D+01i	1.368153376D+03 + 1.495713411D+03i
39	3.850445817D+01 + 4.053358274D+01i	1.267535769D+03 + 1.537154802D+03i
40	4.0241220D+01 + 4.0835416D+01i	1.39246829D+03 + 1.63240898D+03i

TABLE 5 COMPARISON OF EIGENVALUES COMPUTED AT BRANCH POINTS OF MINIMUM MAGNITUDES FOR A VARIETY OF SPHEROIDAL MODES (M, N) WITH THOSE OF DIFFERENT AUTHORS

(m, n)	Oguchi [5]	From [8]	Do-Nhat
(0,0)	1.705180+i4.220186	1.701836+i4.219998	1.705180+i4.220186
(0,1)	10.14083+i11.12158	10.13705+i11.12216	10.14084+i11.12159
(1,1)	2.915318+i6.133951	2.919098+i6.134851	2.915319+i6.133951
(1,2)	12.20109+i16.24407	12.19691+i16.24534	12.20110+ i16.24408
(2,2)	6.102540+i7.684763	6.098946+i7.684379	6.102540 +i 7.684764

TABLE 6 COMPARISON OF CONVERGENCE RADII OF SPHEROIDAL POWER SERIES COMPUTED IN HIGH PRECISION WITH THOSE OF [2] FOR THE SPHEROIDAL SYMMETRIC MODES M = 1

n	From [2]	c <sub>n</sub>   (Do-Nhat)
1,3	4.558875998610D+00	4.558875998610D+00
2,4	5.924714578621D+00	5.924714578621D+00
5	7.359474254879D+00	7.359474254879D+00
6	8.829591878681D+00	8.829591878681D+00
7	1.02592647964D+01	1.02592647964D+01
8	1.165845503426D+01	1.165845503426D+01
9	1.3089161467D+01	1.308916146672D+01
10	1.45421692D+01	1.454216918098D+01
11	1.5971424D+01	1.597142466879D+01
12		1.738071780243D+01
13		1.881049173129D+01
14		2.025644627167D+01
15		2.168554677014D+01
16		2.309980007552D+01
17		2.452922028116D+01
18		2.597132124488D+01
19		2.740035366444D+01
20		2.881756220778D+01
21		3.024681159888D+01
22		3.168648269392D+01
23		3.311548006239D+01
24		3.453465171341D+01
25		3.59638058456D+01
26		3.74018044180D+01
27		3.8830781590D+01
28		4.0251352478D+01
29		4.16804481338D+01
30		4.3117225040D+01
31		4.4546189587D+01
32		4.596780844D+01
33		4.739686565D+01
34		4.883271103D+01
35		5.026166718D+01
36		5.16841003D+01

37		5.31131309D+01
38		5.45482425D+01
39		5.59771928D+01
40		5.74002769D+01

## VI. CONCLUSIONS

The efficient algorithm for the computation of branch points with minimum magnitudes and their corresponding eigenvalues has been presented by using the Newton- Raphson method which relies on the knowledge of the magnitude of the branch point for a given spheroidal mode  $(m,n)$ . Then, the search of the required guess branch points is basically in one dimension in the angular direction from 0 to  $\pi/2$ . This saves a lot of computer time and efficiently allows us to construct tables for these branch points and their associated eigenvalues with a high precision. They are useful for references.

## REFERENCES

- [1] T. Do-Nhat, "Accurate power series for eigenvalues of spheroidal angle functions and their convergence radii," *Can. J. Phys.*, vol. 89, pp. 1083-1099, 2011.
- [2] J. Meixner, F.W. Schafke, and G. Wolf, *Mathieu functions and spheroidal functions and their mathematical foundations*, New York: Springer- Verlag, pp. 102-110, 1980.
- [3] C. Hunter and B. Guerrieri, "The eigenvalues of the angular spheroidal wave equation," *Stud. Appl. Math.*, vol. 66, pp. 217-240, 1982.
- [4] B.E. Barrowes, K. O'Neill, T.M. Grzegorzczuk, and J.A. Kong, "On the asymptotic expansion of the spheroidal wave function and its eigenvalues for complex size parameter," *Stud. Appl. Math.*, vol. 113, pp. 271-301, 2004.
- [5] T. Oguchi, "Eigenvalues of spheroidal wave functions and their branch points for complex values of propagation constants," *Radio. Sci.*, vol. 5, pp. 1207-1214, Aug. 1970.
- [6] T. Barakat, K. Abodayeh and O. Al-Dossary, "The asymptotic iteration method for the angular spheroidal eigenvalues with arbitrary complex size parameter c," *Can. J. Phys.*, vol. 84, pp. 121-129, 2006.
- [7] S. Skokhodov and D. Khristoforov, "Calculation of the branch points of the eigenfunctions corresponding to wave spheroidal functions," *Computation Mathematics and Mathematical Physics*, vol. 46, pp. 1132-1146, 2006.
- [8] L. Li, M. Leong, T. Yeo, P. Kooi, and K. Tan, "Computations of spheroidal harmonics with complex arguments: A review with an algorithm," *Phys. Rev. E*, vol. 58, pp. 6792-6806, Nov. 1998.
- [9] L. Li, X. Kang, and M. Leong, *Spheroidal wave functions in electromagnetic theory*, Wiley, New York, pp. 13-26, 2001.
- [10] C. Flammer, *Spheroidal wave functions*, Stanford, California: Stanford University Press, pp. 16-29, 1957.
- [11] M. Abramowitz and A. Stegun, *Handbook of mathematical functions*, Dover, New York, pp. 751-759, 1970.
- [12] J. Stratton, P. Morse, L. Chu, J. Little, and F. Corbato, *Spheroidal wave functions*, John Wiley and Sons, New York, 1956.
- [13] Morse and Feshbach, *Methods of theoretical physics (Part 2)*, McGraw-Hill, Boston, pp. 642-644, 1953.
- [14] T. Do-Nhat, "Asymptotic expansion of the Mathieu and prolate spheroidal eigenvalues for large parameter c," *Can. J. Phys.*, vol. 77, pp. 635-652, 1999.
- [15] T. Do-Nhat, "Asymptotic expansions of the oblate spheroidal eigenvalues and wave functions for large parameter c," *Can. J. Phys.*, vol. 79, pp. 813-831, 2001.

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## Appendix A

Forward recursion relations for  $n_k, n_{k,x}, n_{k,\lambda}, n_{k,\lambda\lambda}, n_{k,\lambda x}$ 

Note that the forward recursion  $n_k$  is given by (19) with the initial value  $n_1$  of (21).

Taking the partial derivative of  $n_k$  given by (19) with respect to  $x$  yields

$$n_{k,x} = \frac{\partial n_k}{\partial x} = \frac{\delta_{k-1-\lambda}}{\rho_{k-1}x^2} + \frac{\delta_{k-1}}{\rho_{k-1}} \frac{1}{n_{k-1}^2} n_{k-1,x}, k = 2, 3, \dots, \bar{k} \quad (\text{A-1})$$

with

$$n_{1,x} = \frac{\partial n_1}{\partial x} = \frac{\hat{\delta}_0 - \lambda}{\hat{\rho}_0 x^2}, x = c^2 \tag{A-2}$$

Taking the partial derivative of  $n_k$  with respect to  $\lambda$  yields

$$n_{k,\lambda} = \frac{\partial n_k}{\partial \lambda} = \frac{1}{\hat{\rho}_{k-1} x} + \frac{\hat{\sigma}_{k-1}}{\hat{\rho}_{k-1}} \frac{1}{n_{k-1}^2} n_{k-1,\lambda}, k = 2, 3, \dots, \bar{k} \tag{A-3}$$

with

$$n_{1,\lambda} = \frac{\partial n_1}{\partial \lambda} = \frac{1}{\hat{\rho}_0 x} \tag{A-4}$$

Taking the partial derivative of (A-3) with respect to  $\lambda$  yields

$$n_{k,\lambda\lambda} = \frac{\partial^2 n_k}{\partial \lambda^2} = \frac{\hat{\sigma}_{k-1}}{\hat{\rho}_{k-1}} \frac{-2n_{k-1,\lambda}^2 + n_{k-1} n_{k-1,\lambda\lambda}}{n_{k-1}^3}, k = 2, 3, \dots, \bar{k} \tag{A-5}$$

with

$$n_{1,\lambda\lambda} = \frac{\partial^2 n_1}{\partial \lambda^2} = 0. \tag{A-6}$$

Taking the partial derivative of (A-3) with respect to  $x$  yields

$$n_{k,\lambda x} = \frac{\partial^2 n_k}{\partial x \partial \lambda} = -\frac{1}{\hat{\rho}_{k-1} x^2} + \frac{\hat{\sigma}_{k-1}}{\hat{\rho}_{k-1}} \frac{n_{k-1,\lambda x} n_{k-1}^{-2} + n_{k-1} n_{k-1,\lambda\lambda}}{n_{k-1}^3}, k = 2, 3, \dots, \bar{k} \tag{A-7}$$

with

$$n_{1,\lambda x} = \frac{\partial^2 n_1}{\partial x \partial \lambda} = -\frac{1}{\hat{\rho}_0 x^2}. \tag{A-8}$$

### Reverse recursion relations for $n_k, n_{k,x}, n_{k,\lambda}, n_{k,\lambda\lambda}, n_{k,\lambda x}$

First,  $N$  was set such that  $N = \bar{k} + U$  in the program  $U=25$  is sufficient to achieve a high precision accuracy as shown in the above tables. From (28),

$$n_N = c^2 \frac{\bar{D}_N}{\bar{D}_{N-1}} = x \frac{\bar{D}_N}{\bar{D}_{N-1}} \tag{A-9}$$

can be obtained, where

$$\bar{D}_N = \sum_{i=0}^{\infty} \hat{d}_{r, \frac{r}{2}+i} x^i, r = 2N + l \tag{A-10}$$

$$\bar{D}_{N-1} = \sum_{i=0}^{\infty} \hat{d}_{r-2, \frac{r-2}{2}+i} x^i, r = 2N + l. \tag{A-11}$$

Taking the partial derivative of  $\bar{D}_N$  with respect to  $x$  yields

$$\bar{D}_{N,x} = \frac{\partial \bar{D}_N}{\partial x} = \sum_{i=1}^{\infty} i \hat{d}_{r, \frac{r}{2}+i} x^{i-1}, r = 2N + l. \tag{A-12}$$

Similarly,

$$\bar{D}_{N-1,x} = \frac{\partial \bar{D}_{N-1}}{\partial x} = \sum_{i=1}^{\infty} i \hat{d}_{r-2, \frac{r-2}{2}+i} x^{i-1}, r = 2N + l. \tag{A-13}$$

is obtained. Therefore, the partial derivative of  $n_N$  with respect to  $x$  is given by

$$n_{N,x} = \frac{\bar{D}_N}{\bar{D}_{N-1}} + x \left[ \frac{1}{\bar{D}_{N-1}} \bar{D}_{N,x} - \frac{\bar{D}_N}{\bar{D}_{N-1}^2} \bar{D}_{N-1,x} \right]. \tag{A-14}$$

Since  $n_N$  given by (A-9) is a function of  $x$ , the partial derivatives of  $n_N$  with respect to  $\lambda$  are zero. Therefore,

$$n_{N,\lambda} = \frac{\partial n_N}{\partial \lambda} = 0 \tag{A-15}$$

$$n_{N,\lambda\lambda} = \frac{\partial^2 n_N}{\partial \lambda^2} = 0 \tag{A-16}$$

$$n_{N,\lambda x} = \frac{\partial^2 n_N}{\partial x \partial \lambda} = 0. \tag{A-17}$$

The reverse recursion relation given by (22)

$$n_k = -\frac{\hat{\sigma}_k x}{\hat{\rho}_k x n_{k+1} + \hat{\delta}_k + \zeta_k x - \lambda}, k = N - 1, N - 2, \dots, \bar{k} + 2 \tag{A-18}$$

is recursively calculated from  $n_N$  given by (A-9) from which  $n_{\bar{k}+2}$  can be computed.

By differentiating (A-18) with respect to  $x$ ,

$$n_{k,x} = \frac{\partial n_k}{\partial x} = -\hat{\sigma}_k \frac{\hat{\delta}_k - \lambda - x^2 \hat{\rho}_k n_{k+1,x}}{(\hat{\rho}_k x n_{k+1} + \hat{\delta}_k + \hat{\zeta}_k x - \lambda)^2}, k = N-1, N-2, \dots, \bar{k}+2 \quad (\text{A-19})$$

can be obtained. Hence,  $n_{\bar{k}+2,x}$  is recursively calculated from  $n_N$  given by (A-9) and  $n_{N,x}$  given by (A-14).

By differentiating (A-18) with respect to  $\lambda$ ,

$$n_{k,\lambda} = \frac{\partial n_k}{\partial \lambda} = \frac{x \hat{\sigma}_k (\hat{\rho}_k x n_{k+1, \lambda} - 1)}{(\hat{\rho}_k x n_{k+1} + \hat{\delta}_k + \hat{\zeta}_k x - \lambda)^2}, k = N-1, N-2, \dots, \bar{k}+2 \quad (\text{A-20})$$

Hence,  $n_{\bar{k}+2}$  is recursively calculated from  $n_N$  given by (A-9) and  $n_{N,\lambda} = 0$  given by (A-15).

By differentiating (A-20) with respect to  $\lambda$ ,

$$n_{k,\lambda\lambda} = \frac{\partial^2 n_k}{\partial \lambda^2} = \frac{\partial n_{k,\lambda}}{\partial \lambda} = \frac{\hat{\sigma}_k x}{(\hat{\rho}_k x n_{k+1} + \hat{\delta}_k + \hat{\zeta}_k x - \lambda)^2} \left[ \hat{\rho}_k x n_{k+1,\lambda\lambda} - 2 \frac{(\hat{\rho}_k x n_{k+1, \lambda} - 1)^2}{\hat{\rho}_k x n_{k+1} + \hat{\delta}_k + \hat{\zeta}_k x - \lambda} \right], k = N-1, \dots, \bar{k}+2 \quad (\text{A-21})$$

Hence,  $n_{\bar{k}+2,\lambda\lambda}$  is recursively calculated from  $n_N$ ,  $n_{N,\lambda} = 0$ , and  $n_{N,\lambda\lambda} = 0$  given by (A-9), (A-15) and (A-16), respectively.

Finally, by differentiating (A-19) with respect to  $\lambda$

$$n_{k,x\lambda} = n_{k,\lambda x} = \frac{\partial^2 n_k}{\partial \lambda \partial x} = \frac{\partial n_{k,x}}{\partial \lambda} = \frac{\hat{\sigma}_k}{(\hat{\rho}_k x n_{k+1} + \hat{\delta}_k + \hat{\zeta}_k x - \lambda)^2} \left[ \hat{\rho}_k x^2 n_{k+1,x\lambda} + 1 + 2 \frac{(-1 + \hat{\rho}_k x n_{k+1, \lambda})(\hat{\delta}_k - \lambda - \hat{\rho}_k x^2 n_{k+1,x})}{\hat{\rho}_k x n_{k+1} + \hat{\delta}_k + \hat{\zeta}_k x - \lambda} \right], \quad (\text{A-22})$$

$$k = N-1, N-2, \dots, \bar{k}+2$$

Hence,  $n_{\bar{k}+2,x\lambda}$  is recursively calculated from  $n_N$ ,  $n_{N,\lambda} = 0$ ,  $n_{N,x}$  and  $n_{N,x\lambda} = 0$ , as given by (A-9), (A-15), (A-14) and (A-17), respectively.